

Dynamic Equilibrium with Overpriced Put Options

Abstract

It is a well-known anomaly that prices of put options are too high when options are out-of-the-money. This paper presents a simple general equilibrium model of the market where European put options become substantially overpriced when they are out-of-the-money. Overpricing is due to the presence of short-sale constraints on trading stocks and derivatives, as well as the heterogeneity between investors. We confirm the predicting power of the model by comparing its implications with existing empirical results.

Keywords: Derivatives; Equilibrium; Arbitrage

JEL Classification: D52, G12, G13

1 Introduction

It is well known that the European options market does not follow the Black–Scholes model. This fact is revealed by the implied volatility of a stock which increases when a put option goes out-of-the-money (OTM) or a call option goes in-the-money (ITM). Moreover, it has been well documented that overpricing is much more pronounced for the put option than for the call option in both the American and European types. Table 1 reports the comparison statistics for put and call options written on the S&P 500 futures at different moneyness. As seen, the scale of return is much higher for OTM puts than for OTM calls. Furthermore, the Sharpe ratio of the put option increases in magnitude a few times when the put goes OTM. Several recent papers have reported similar results. For example, Jackwerth (2000), Coval and Shumway (2001), Bakshi and Kapadia (2003) analyzed the pricing of put options. These papers considered different data sets and trading strategies to conclude that OTM put options are historically too expensive.¹ The fact that put options are overpriced so greatly is puzzling, and it is important to explain it within the general equilibrium framework.

Recently, Bondarenko (2003) tested a wide range of different equilibrium models for generating the mispricing of puts by using a “model-free” methodology. He considered three different explanations for the puzzle: 1) High prices for puts are expected and reflect normal risk premiums under some equilibrium model of the market; 2) Investors have inaccurate beliefs; 3) The sample under investigation is affected by the Peco problem. In these explanations, the pricing kernel in the market cannot be path-dependent, but the underlying process may have jumps and the market could be dynamically incomplete and have frictions. Bondarenko concluded that none of the explanations is supported by the

¹Similar results can be found in many papers, including Bates (2000), Buraschi and Jackwerth (2001), Ait-Sahalia, Wang, and Yared (2001), Branger and Schlag (2002), Pan (2002), Rosenberg and Engle (2002), Amin, Coval, and Seyhun (2003), Bollen and Whaley (2004), and Han (2005).

data. Therefore, the pricing kernel in the economy should be path-dependent, or should not exist at all. Notice that the ITM call overpricing could be derived from at least one of the explanations considered by Bondarenko. In a related paper, Driessen and Maenhout (2003) take a note that numerous investors have long positions in overpriced put options, and argue that it is possible only with highly distorted probability assessments.²

To the best of our knowledge, there exist three models generating the put price anomaly in the general equilibrium framework. All these models rely on the existence of the stock crashes. The model developed by Bates (2001) assumes path-dependent preferences which explicitly depend on crashes of the exogenous dividend process. Trading the put option dynamically completes the market, and an investor's aversion to jumps generates additional risk premium for writing put options which are vary sensitive to falls in the stock. Liu, Pan, and Wang (2005) use the robust control approach, assuming that a representative agent does not know the true model of rare events and follows the worst-case scenario in choosing the right one. The resulting pricing kernel is similar to that of Bates' model. Finally, the model by Benzoni, Collin-Dufresne, and Goldstein (2005) uses a representative agent with Epstein-Zin preferences and assumes that the aggregate dividend is driven by a persistent stochastic growth variable that can jump. Put overpricing is generated by a high elasticity of intertemporal substitution (of the order of 2) of a representative agent. In their framework, the risky asset performs purely in a bad state and investors are willing to pay a high price for a security that provides insurance against this state.

In this paper we propose a simple general equilibrium model of the market that generates significant overpricing of at-the-money (ATM) and especially OTM put options. We

²Related results can also be found in Santa-Clara and Saretto (2005), Santa-Clara and Yan (2005), and Garleanu, Pedersen, and Poteshman (2005).

Table 1: Monthly Characteristics of Puts, Calls and S&P 500 Futures

	k						
	0.94	0.96	0.98	1.00	1.02	1.04	1.06
Mean of put returns	-0.95	-0.58	-0.54	-0.39	-0.26	-0.17	-0.12
Sharpe ratio of put	-3.93	-0.38	-0.40	-0.35	-0.28	-0.24	-0.18
Mean of call returns	0.06	0.06	0.05	0.04	-0.04	-0.06	0.21

The table reports monthly returns for puts and calls written on S&P 500 futures as well as the Sharpe ratio of puts. Statistics are reported with different moneyness k . The mean monthly return and the Sharpe ratio of the S&P 500 futures are 0.007 and 0.18, respectively. These results are borrowed from Tables 1 and 2 of Bondarenko (2003).

consider a pure-exchange competitive financial economy which includes a locally riskless bond, a stock, and a derivative written on the stock. The stock market is bullish. There are two agents, one who could be associated with an individual investor, and the other who could be related to an institutional investor. One investor has full knowledge of the underlying process, while the other has incomplete information and receives pessimistic shocks during some states of the economy. In addition, each investor has to satisfy the short-sale constraints on the stock and the derivative. The inability of agents to fully adjust their asset allocations to systematic shocks creates the put overpricing. The idea of the model is taken from Basak and Croitoru (2000)³. In their important paper, BC show that the presence of constraints on trading a stock and its derivative, as well as the heterogeneity between agents, can cause the absence of a pricing kernel, which can create substantial mispricing of a derivative. These authors assume the option return volatility to be exogenous and consider constraints limiting a long position in the derivative and prohibiting short-selling of the stock. Naturally, the approach taken by BC disregards the feedback effects on the option return volatility. Thus, their analysis is the partial equilibrium one. Such analysis cannot establish if there would be states with mispricing in the economy with a given type of a derivative. In our treatment

³Henceforth BC

we evaluate the feedback effects between all interdependent markets in the economy and establish the existence of states with mispricing.⁴ The extension of the BC analysis to a general equilibrium one is crucial for explaining the put overpricing and is an additional contribution of this paper.

The importance of investors' beliefs in option pricing has recently been confirmed empirically by Han (2005).⁵ Han shows that the index option smile strongly depends on the aggregate sentiments of investors. We add to this result by demonstrating how the heterogeneity in sentiments affects the steepness of the volatility smile implied by put options. We find that there is a strong overpricing of ATM and OTM put options at states where an individual investor receives pessimistic shocks. At these states, the instantaneous market price of risk of the put becomes a few times greater than that of the underlying asset. Moreover, the magnitude of the conditional returns of the put also rises a few times with respect to that of the put calculated in the assumption that the instantaneous market prices of risk of two risky assets are equal.

It is likely that put option prices recorded in the market were generated during many different states of the economy. Hence, we consider the effect of overpricing averaged over shocks received by a pessimistic investor. In particular, we assume that this investor's beliefs are correct or that he thinks the stock is undergoing the 1987 stock market crash. We find that in order for the model to generate the average monthly returns similar in magnitude to that in Table 1, a pessimistic investor must expect the 1987 stock market

⁴The difference between the analysis in the partial and general equilibrium frameworks is often not significant. For example, the economy where the bond market is treated exogenously and the stock market is treated endogenously. The rationale behind this approach is that the stock market has rather small effects on the interest rate and the feedback effects are insignificant. The situation is different in our market. The price of the option (and so its volatility) is defined by the stock and the bond markets. Thus, endogenous consideration of the option volatility is important in understanding the properties of asset prices and is critical in verifying the existence of the equilibrium with mispricing.

⁵Also see Shefrin (1999), and Buraschi and Jiltsov (2003).

crash to occur approximately every thirteen months! Therefore rendering a conclusion in support of behavioral finance. We notice that the absence of short–sale constraints and thus, the put overpricing, results in the expected returns of the put to be a few times lower than those observed. Moreover, the predictions of our model do not contradict existing market practice when many investors take long positions in both risky securities to achieve portfolio insurance. Our explanations agree with recent empirical results by Han (2005), who shows that investors’ sentiments are an important determinant of option prices in the presence of limits to arbitrage.

In contrast with the models by Bates (2001), Liu, Pan, and Wang (2003), and Benzoni, Collin–Dufresne, and Goldstein (2005), our model underlines the importance of short–sale constraints on risky securities for explaining put overpricing. Moreover, differently from the first two models, we assume simple, easy–interpreted CRRA preferences. In addition, our model naturally explains the significant difference in magnitudes of Sharpe ratios of the two risky securities. Furthermore, the underlying process for the stock index does not have to include crashes for our model to generate put overpricing. However to match the data quantitatively and make the interpretation of the results clearer, we will assume that a pessimistic investor may believe in stock market crashes.

The rest of the article is organized as follows. Section 2 describes the model. Section 3 presents results of optimal control problems faced by agents. Section 4 derives the equilibrium with the different instantaneous market prices of risk of two risky securities. Section 5 presents the output of the model for put prices, as well as their conditional returns and Sharpe ratios. Section 6 summarizes. Appendix A describes bayesian learning of μ_D . The proof of Lemma 3 is in Appendix B. Appendix C treats the economy with more than one option.

2 Economic Setting

We begin with an economic setting that is similar to that in BC. We consider a pure-exchange economy with a finite horizon T , where there is a single perishable consumption good that we treat as the numeraire.

2.1 Information structure

Uncertainty is represented by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$ on which a two-dimensional standard Brownian motion $W = (w, b)$ is defined. Brownian motions w and b are independent. Let us then denote by $\{\mathcal{F}_t^W\}$ the augmented filtration generated by W , and by \mathcal{H} a σ -field independent of $\{\mathcal{F}_t^W\}$. The complete filtration $\{\mathcal{F}_t\}$ is the augmentation of the filtration $\mathcal{H} \times \{\mathcal{F}_t^W\}$.

There are two agents, defined by 1 and 2, who both have logarithmic preferences. They observe the exogenous positive dividend rate D which follows the diffusion process

$$dD(t) = D(t)[\mu_D(t)dt + \sigma_D dw(t)], \quad (1)$$

where σ_D is a positive constant. The conditional growth rate $\mu_D(t)$ is $\{\mathcal{F}_t\}$ -progressively measurable and follows the mean-reverting process

$$d\mu_D(t) = [a_0 + a_1\mu_D(t)]dt + d db(t), \quad (2)$$

where a_0, d are some positive constants and a_1 is a negative constant. Moreover, $\mu_D(0)$ is \mathcal{H} -measurable.

Agent 1 has full information about process D , so his beliefs are given by the probability measure Q^1 that is identical to Q . Meanwhile, agent 2 may have an incorrect view of process

D . This agent has the information represented by the filtration $\{\mathcal{F}_t^D\} \subset \{\mathcal{F}_t\}$, $t \in [0, T]$, where $\{\mathcal{F}_t^D\}$ denotes the filtration generated by $D(t)$. He deduces σ_D from the quadratic variation of D , but can only draw inferences about μ_D . Agent 2 beliefs are represented by a probability measure Q^2 that is equivalent to the original probability measure Q on \mathcal{F} . Q^2 and Q may disagree on \mathcal{H} , meaning that agent 2 has incorrect prior beliefs. This agent updates his beliefs about μ_D in a bayesian fashion, so that his estimate of μ_D at time t is given by $\mu_D^2(t) = E^2[\mu_D | \mathcal{F}_t^D]$, where $E^2[\cdot]$ denotes the expectation relative to Q^2 . Appendix A gives more details about inferencing μ_D^2 . Agent 2 may receive optimistic or pessimistic shocks in his beliefs but only pessimistic shocks are important to our model and so we call him a pessimist.

The innovation process w^2 induced by agent 2's beliefs and filtration, is given by

$$dw^2(t) = \sigma_D^{-1}(t) \left(\frac{dD(t)}{D(t)} - \mu_D^2(t) dt \right) = dw(t) + \bar{\mu}_D(t) dt, \quad (3)$$

where $\bar{\mu}_D(t) = \sigma_D^{-1}(t)(\mu_D(t) - \mu_D^2(t))$.

It is known from the filtering theory that the innovation process w^2 is a Q^2 -Brownian motion on $\{\mathcal{F}_t^{w^2}\}$ which is a filtration generated by w^2 . Moreover, agent 2's information and innovation filtration coincide with those generated by D , $\{\mathcal{F}_t^D\} = \{\mathcal{F}_t^{w^2}\} \equiv \{\mathcal{F}_t^2\}$.

Finally, because agent 1 has correct beliefs, we define $\mu_D^1 \equiv \mu_D$, $w^1 \equiv w$, and $\{\mathcal{F}_t^1\} \equiv \{\mathcal{F}_t\}$.

2.2 Securities

We assume that agents can continuously trade three securities: a riskless bond, a stock that represents an index, and a European put option written on the stock. The riskless bond pays no dividends, and its price has the following dynamics:

$$dB(t) = B(t)r(t)dt, \quad (4)$$

where r denotes the interest rate process.

The stock is in constant net supply of 1, and it pays a continuous dividend at the rate of $D(t)$. The price of the stock $S(t)$ is a unique strong solution of the stochastic differential equations

$$dS(t) + D(t)dt = S(t)[\mu_S(t)dt + \sigma_S(t)dw(t)] = S(t)[\mu_S^2(t)dt + \sigma_S(t)dw^2(t)], \quad (5)$$

where $\mu_S^2(t) = E^2[\mu_S(t)|\mathcal{F}_t^D]$, and we require that $\sigma_S(t) > 0$ for $t \in [0, T)$.

The derivative is in zero net supply, has a strike price of K and expires on day $T^* < T$. The price of the derivative $P(t)$ is a unique strong solution of the stochastic differential equations

$$dP(t) = P(t)[\mu_P(t)dt + \sigma_P(t)dw(t)] = P(t)[\mu_P^2(t)dt + \sigma_P(t)dw^2(t)], \quad (6)$$

where $\mu_P^2(t) = E^2[\mu_P(t)|\mathcal{F}_t^D]$, and we require that $\sigma_P(t) < 0$ for $t \in [0, T^*)$.

Price agreement across agents implies the following relationships between the drifts of returns perceived by two agents:

$$\bar{\mu}_D(t) = \bar{\mu}_\alpha(t), \quad \bar{\mu}_\alpha(t) = \sigma_\alpha^{-1}(t)(\mu_\alpha(t) - \mu_\alpha^2(t)), \quad \alpha \in \{P, S\}. \quad (7)$$

Letting π_S^n and π_P^n denote the proportions of agent n 's wealth X^n invested in the stock and the option, respectively, we assume that at all times $t \in [0, T]$ each agent faces short position constraints in holdings of both risky securities⁶:

$$-\beta \leq \pi_S^n, \quad -\gamma \leq \pi_P^n, \quad (8)$$

where β is a nonnegative constant and γ is a nonnegative function of the stock and the put option prices. In particular, we set $\gamma = \min\{\frac{P}{0.15S - (S-K)^+ + P}, \frac{P}{0.1K + P}\}$. This choice of the short sale constraint on the put option is motivated by the existing market practice of

⁶At $t > T^*$, $\pi_P^n = 0$ and the constraint on option holdings is trivially satisfied.

setting margins on naked put options written on index (see Hull (2002)). According to this practice, an investor writing a naked put on the index is required to deposit the proceeds received plus the greater of 15% of the underlying less out-of-the money or 10% of the exercise price of the put. Thus, an investor must have a portfolio with a value of at least $\max\{0.15S - (S - K)^+ + P, 0.1K + P\}$ dollars to be able to short the put.⁷ If the put is covered, then the required positions in the bond and the stock make γ even smaller than suggested.

With our choice of the constraints on a trading portfolio, we consider only the short-sale ones and neglect those on long positions in risky assets. However, the short-sale constraints are typically much tighter than those on long positions, and thus have the strongest effect on asset prices. Additionally, we set the bounds β and γ to be independent from the type of investor. This may seem to contradict the existing practice. For example, an investment bank can often take a very short position in the stock, while shorting the stock by an individual investor is frequently prohibited. However, as will be shown in Section 4, when the constructed market is in equilibrium, agents' allocations are either not binding or each constraint is binding for one of two agents. When the constraints are binding, agent 1 takes a long position in the stock and a short one in the put option, while agent 2 takes a long position in the put option and a short one in the stock. It follows that in our model the important constraints are that faced by a pessimist on shorting the stock and that faced by a realist on shorting the put option. Thus, we can safely assume that the short-sale constraints are different across agents.

We expect that the risky assets' instantaneous market prices of risk could be different,

⁷The constraints could be due not only to market regulations but also to the volunteer commitment of an investor. For example, Chen et al. (2002) argue that many mutual funds still abide by the restrictions of the Investment Company Act of 1940, even though they are no longer binding. In fact, Almazan et al. (2002) show that only a small fraction of mutual funds shorts stocks.

and we characterize this difference by the process

$$\Delta(t) \equiv \frac{\mu_P(t) - r(t)}{\sigma_P(t)} - \frac{\mu_S(t) - r(t)}{\sigma_S(t)}. \quad (9)$$

It follows from equation (7) that agents agree on the value of Δ , that is,

$$\Delta(t) = \Delta^2(t).$$

If the market allows states with different instantaneous market prices of risk, then it may admit states with arbitrage opportunities for a price-taking speculator. Moreover, because σ_S is always positive, the market that includes only the bond and the stock does not allow arbitrage opportunities.

Because the risky assets' instantaneous market prices of risk could be different, each risky security has its own security-specific deflator process ξ_j^n , that also differs across agents and follows a diffusion process

$$d\xi_j^n(t) = -\xi_j^n(t) \left[r(t)dt + \left(\frac{\mu_j^n(t) - r(t)}{\sigma_j(t)} \right) dw^n(t) \right], \quad j \in \{S, P\}, \quad (10)$$

where $\mu_j^1 \equiv \mu_j$.

From the dynamics of the prices and the last expression, it follows that each deflated security gains process is a Q^n -martingale. In particular, $\xi_P^n P$ is a Q^n -martingale, leading to the present value formula:

$$P(t) = \frac{1}{\xi_P^n(t)} E_t^n[\xi_P^n(T^*)(K - S_{T^*})^+], \quad n \in \{1, 2\}, \quad \forall t \in [0, T^*]. \quad (11)$$

We will use equation (11) to determine the price of the derivative after finding its deflator from the market clearing conditions.

Notice that the dividend rate in our model has a continuous path, while in practice it is susceptible to crashes. For example, one may expect that this rate (and so the stock price

to be defined later) undergoes crashes of the 1987 stock crash scale approximately every 50 years. Incorporation of such crashes will decrease the long positions in the stock by agent 1, but not enough to effect our conclusions. Therefore, we exclude crashes for reasons of computational simplicity. More generally, one may consider an economy with crashes where there are pessimistic and optimistic agents learning about the drift of the dividend rate, as well as frequency and size of its crashes. The conclusions of such an analysis and this paper will be the same.

It is interesting to relate agent 1 and agent 2 to existing types of investors. One can assume that agent 1 represents an institutional investor and/or a market maker who has a fairly good knowledge of markets, while agent 2 stands for an individual investor who may have an incorrect knowledge of markets. Even though such an affiliation is reasonable in many situations, we will not use it to avoid contradictions with some empirical observations.⁸

Two remarks remain to follow. First, our model requires the presence of a pessimistic agent. Despite the fact that this agent is bayesian, it is essential that disagreement across agents persists through time. On the other hand, the existing overpricing of put options has been documented at least since 1987, so that one can argue that a bayesian agent should have become realistic by now. This, however, does not have to be the case. For example, during the extensive stock market growth of 1990s there was substantial disagreement over its economic nature. From one point of view, the growth was expected and was due to new economic fundamentals, while from the other the stock market was substantially overpriced and would crash. This divergence persisted for many years. On a theoretical level, we notice that contrary to our assumption, the true functional form of the dividend is

⁸For example, if pessimistic shocks are weak, then the put overpricing requires tight constraint on the stock short-sale (e.g., $\beta = 0$). While in many cases an individual investor has this constraint rather tight, in some cases an institutional investor does. These facts can lead to confusion, because agent 2 takes a short position in the stock when markets are in equilibrium.

unknown to anyone. Moreover, this form changes over time and investors develop only its approximate estimates. As a result, uniform agreement on the dividend rate across agents can never develop. Furthermore, the pessimistic agent does not have to have a unique physical affiliation through the time. In particular, agent 2 could be a group of investors who received a pessimistic shock three months ago. It has improved its knowledge of the stock return by now, but there would be another group of investors which will receive such a shock tomorrow. The groups could be very small if the shocks are very strong (see discussion in Section 5). Finally, we notice that if agent 2 believes in stock market crashes, then he can learn about their sizes only during their occurrences. Given that crashes are very rare, the pessimistic agent can update his knowledge very slowly and the misspecification of the dividend rate drift persists for a very long time.⁹ Thus, rationale behind the long-term presence of agent 2. Second, we consider a general equilibrium model where all agents are logarithmic. Logarithmic preferences render the stock price to be a function of only time and the dividend rate, thus excluding any effects from the presence of constraints. Instead, these effects are exaggerated in the bond price. Consequently, the interest rate exhibits a high variability that is not observed in the data. It would be more rewarding to consider equilibrium with an exogenous riskless technology, so that the effects of the constraints are neglected in the interest rate, but are fully evaluated in the stock price. Nevertheless, we do not choose this approach because of its computational complexity.

Overall, we exploit an economic mechanism which is similar to that in BC, and naturally use a number of their assumptions for tractability of calculations. However, the analysis of BC cannot explain the OTM put overpricing because the explanation requires the short sale constraint on the put option to be dependent on its moneyness, and uses a general

⁹See also Benzoni, Collin-Dufresne, and Goldstein (2005).

equilibrium approach allowing evaluation of the feedback effects on the option price.

3 Agents' Optimal Consumptions and Policies

We assume that agent n , $n \in \{1, 2\}$, is endowed at time 0 with e^n shares of the stock ($e^n > 0$, $e^1 + e^2 = 1$), providing him with initial wealth $X^n(0) = e^n S(0)$. A nonnegative consumption process c^n and a trading strategy π^n are chosen by agents 1 and 2 from the set of $\{\mathcal{F}_t\}$ and $\{\mathcal{F}_t^D\}$ -progressively measurable processes, respectively. A pair of consumption and portfolio processes $(c^n, \pi^n) \in \mathcal{L}_+ \times \mathcal{L}^2[0, T]$ for agent n is admissible, if it satisfies the portfolio constraints in (8) and the nonnegativity restriction $X^n(t) \geq 0$, $t \in [0, T]$, where the agent's wealth X^n satisfies the stochastic differential equation

$$\begin{aligned} dX^n(t) &= -c^n(t)dt + X^n(t) \left\{ [r(t) + \pi_S^n(t)(\mu_S^n(t) - r(t)) + \pi_P^n(t)(\mu_P^n(t) - r(t))]dt \right. \\ &\quad \left. + [\pi_S^n(t)\sigma_S(t) + \pi_P^n(t)\sigma_P(t)]dw^n(t) \right\}, \quad X^n(0) = e^n S(0). \end{aligned} \quad (12)$$

An agent maximizes his time-additive, state-independent logarithmic utility

$$\max_{c^n} E^n \int_0^T \log(c^n(t))dt, \quad (13)$$

over all admissible pairs (c^n, π^n) subject to the dynamic budget constraint (12).

Now let us consider an agent's utility maximization when Δ could be nonzero. Similar to BC, we will handle an agent's optimal strategy by introducing the risk-weighted sum of holdings

$$\Phi^n \equiv \pi_S^n + \frac{\sigma_P}{\sigma_S} \pi_P^n. \quad (14)$$

Instead of making decisions on allocations between the two risky securities, agent n will decide how much to allocate in this composite risky asset.

¹⁰If $t > T^*$ then all the above definitions apply, if we set π_P equals zero.

We first notice that states of economy with negative Δ cannot exist in equilibrium. Indeed, let us assume that $\pi_P^n > 0$ and $\pi_S^n = -\frac{\sigma_P}{\sigma_S}\pi_P^n > 0$. Then equation (12) becomes

$$\begin{aligned} dX^n &= -c^n dt + X^n \left\{ [r + \Phi^n(\mu_S^n - r) + \pi_P^n \sigma_P \Delta] dt + \Phi_S^n \sigma_S dw^n \right\} \\ &= -c^n dt + X^n (r + \pi_P^n \sigma_P \Delta) dt. \end{aligned}$$

Given that π_P^n and π_S^n are not bounded from above, an investor would like to make them, and so the drift of his wealth, arbitrary large. Thus, the markets will never clear and Δ could only be positive or zero in the equilibrium. It follows that the option price can be equal or higher than that found in the assumption that $\Delta = 0$.

The following lemma describes the optimal allocations of an agent when the positive difference in the instantaneous market prices of risk is possible. This lemma is a straightforward modification of the results by BC for the case of the short-sale constraints and is presented without proof.

Lemma 1 *Suppose that Δ could be positive or zero. If $t \leq T^*$ then agent n 's optimal portfolio holdings and composite investment are as in Table 2. If $t > T^*$ then $\Phi^n = \pi_S^n = \frac{\mu_S^n - r}{\sigma_S^n}$. $\forall t < T$ his rate of consumption is given by*

$$c^n(t) = \frac{X^n(t)}{T - t}. \quad (15)$$

Table 2 reports the four cases that agent n may face in optimally allocating risk between the securities S and P when $t \leq T^*$: in case (a) there is no difference in the instantaneous market prices of risk, but it exists in cases (b)–(d).

Table 2: Optimal allocations

Case	Conditions	Δ	π_S^n	π_P^n	Φ^n
(a)	$\frac{\mu_S^n - r}{\sigma_S} = \frac{\mu_P^n - r}{\sigma_P}$	0	$> -\beta$	$> -\gamma$	$\frac{\mu_S^n - r}{\sigma_S^2}$
(b)	$\frac{\mu_S^n - r}{\sigma_S} \geq -(\sigma_S \beta + \sigma_P \gamma)$	> 0	$\frac{\mu_S^n - r}{\sigma_S^2} + \gamma \frac{\sigma_P}{\sigma_S}$	$-\gamma$	$\frac{\mu_S^n - r}{\sigma_S^2}$
(c)	$\frac{\mu_S^n - r}{\sigma_S} < -(\sigma_S \beta + \sigma_P \gamma) < \frac{\mu_P^n - r}{\sigma_P}$	> 0	$-\beta$	$-\gamma$	$-\beta - \gamma \frac{\sigma_P}{\sigma_S}$
(d)	$-(\sigma_S \beta + \sigma_P \gamma) \geq \frac{\mu_P^n - r}{\sigma_P}$	> 0	$-\beta$	$\frac{\mu_P^n - r}{\sigma_P^2} + \beta \frac{\sigma_S}{\sigma_P}$	$\frac{\mu_P^n - r}{\sigma_P \sigma_S}$

The table reports the optimal portfolio holdings of an agent n who may face the four cases (a)–(d) in optimally allocating risk between two securities S and P .

4 Equilibrium

Definition 1 *An equilibrium is a price system $(\mu_S, \mu_S^2, \sigma_S, \mu_P, \mu_P^2, \sigma_P)$ and admissible consumption–portfolio processes (c^n, π^n) , $n = 1, 2$ such that (i) agents choose their optimal consumption–portfolio strategies given their knowledge of price processes in $(\Omega, \mathcal{F}, \{\mathcal{F}_t^n\}, \mathcal{Q}^n)$; (ii) security prices are consistent across agents, that is,*

$$\bar{\mu}_D(t) = \sigma_\alpha^{-1}(t)(\mu_\alpha(t) - \mu_\alpha^2(t)), \quad \alpha \in \{P, S\}$$

and (iii) good and security markets clear, that is $\forall t \in [0, T]$

$$c^1(t) + c^2(t) = D(t), \quad (16)$$

$$\pi_S^1(t)X^1(t) + \pi_S^2(t)X^2(t) = S(t), \quad (17)$$

$$\pi_P^1(t)X^1(t) + \pi_P^2(t)X^2(t) = 0, \quad (18)$$

$$X^1(t) + X^2(t) = S(t). \quad (19)$$

Because the put mispricing can exist only when the put is available in the economy we are interested in the equilibrium when $t \leq T^*$. Agent n can face one of four different cases (a), (b), (c) or (d) of Lemma 1. In general equilibrium with two agents, the market will clear only in the two configurations: (a, a) and (b, d). Indeed, if one agent does not see the

difference in the instantaneous market prices of risk, neither does the other. Therefore, if one agent is in case (a), then so is the other. If $\Delta > 0$ then both agents cannot be in the same case, since at least one of the markets cannot otherwise clear. For the same reason, none of the agents can be in case (c). If agent 2 experiences a pessimistic shock he will be in case (d) and the first agent will be in case (b) when $\Delta > 0$. Lastly, there could be states where agent 2 becomes optimistic and the configuration (d, b) occurs. However, the probability of this outcome is negligibly small, because it requires stock returns to be falling at a very high rate. Thus, we can neglect the outcome (d, b), and only the configuration (b, d) allowing the difference in the instantaneous market prices of risk will be considered.

Let us denote the ratio of wealth of the two agents $X^2(t)/X^1(t)$ as λ . By using market clearing conditions, one can characterize the equilibrium in terms of the volatility of the derivative return σ_P (so that this equilibrium is partial). The following result is a modification of Proposition 2 in BC for the case of short-sale constraints, and is presented without proof.

Lemma 2 *Assume that the stochastic differential equation for λ given by equations (24) and (28), with initial condition $\lambda(0) = e^2/e^1$, has a strictly positive solution and the optimal polices and prices satisfy the technical conditions of Sections 2.2 and 3.*

Let $t \leq T^$, then the partial equilibrium exists, and the equilibrium consumption allocations, stock price, expected growth rate and volatility are given by*

$$c^1(t) = \frac{D(t)}{1 + \lambda(t)}, \quad c^2(t) = \frac{D(t)\lambda(t)}{1 + \lambda(t)}, \quad (20)$$

$$S(t) = (T - t)D(t), \quad \mu_S^n(t) = \mu_D^n(t), \quad n \in \{1, 2\}, \quad \sigma_S(t) = \sigma_D. \quad (21)$$

An equilibrium could be in one of two configurations, characterized by the difference in the instantaneous market prices of risk $\bar{\mu}_D(t)$ and the stochastic weighting $\lambda(t)$.

When

$$\frac{\lambda(t) + 1}{\lambda(t)} [\sigma_D \lambda(t) (\beta + 1) - \sigma_P(t) \gamma(t)] < \bar{\mu}_D(t) \quad (22)$$

agents are in (b, d) and the difference in the instantaneous market prices of risk, the stochastic differential equation for λ , the interest rate and the risky allocations for each agent are

$$\Delta(t) = \bar{\mu}_D(t) - \frac{1 + \lambda(t)}{\lambda(t)} [(\beta + 1) \sigma_D \lambda(t) - \gamma(t) \sigma_P(t)], \quad (23)$$

$$\frac{d\lambda(t)}{\lambda(t)} = \Delta(t) \gamma(t) \sigma_P(t) \frac{1 + \lambda(t)}{\lambda(t)} dt + (\Delta(t) - \bar{\mu}_D(t)) dw(t), \quad (24)$$

$$r(t) = \mu_D^1(t) - \sigma_D^2 - \sigma_D [\sigma_D (\beta + 1) \lambda(t) - \gamma(t) \sigma_P(t)], \quad (25)$$

$$\pi_S^1(t) = \frac{\mu_S^1(t) - r(t)}{\sigma_D^2(t)} + \gamma(t) \frac{\sigma_P(t)}{\sigma_D}, \quad \pi_P^1(t) = -\gamma(t), \quad (26)$$

$$\pi_S^2(t) = -\beta, \quad \pi_P^2(t) = \frac{\mu_P^2(t) - r(t)}{\sigma_P^2(t)} + \beta \frac{\sigma_D}{\sigma_P(t)}. \quad (27)$$

When inequality (22) does not hold, then agents are in (a, a) and the stochastic differential equation for λ , the interest rate and the cumulative risk exposure of each agent are

$$\frac{d\lambda(t)}{\lambda(t)} = -\bar{\mu}_D(t) dw(t), \quad (28)$$

$$r(t) = \mu_D^1(t) - \frac{\lambda(t) \bar{\mu}_D(t) \sigma_D}{1 + \lambda(t)} - \sigma_D^2, \quad (29)$$

$$\Phi^1(t) = 1 + \frac{\lambda(t) \bar{\mu}_D(t)}{\sigma_D (1 + \lambda(t))}, \quad \Phi^2(t) = 1 - \frac{\bar{\mu}_D(t)}{\sigma_D (1 + \lambda(t))}. \quad (30)$$

In the above configuration (a, a), $\Delta = 0$ and allocations for each risky security cannot be identified uniquely.

If $t > T^*$, then the equilibrium consumption allocations, stock price, expected growth rate and volatility are given by equations (20) and (21), while the stochastic differential equation for λ , the interest rate and the cumulative risk exposure of each agent are given by equations (28)–(30), where $\Phi^n = \pi_S^n$. Here μ_P , μ_P^2 , and σ_P are not defined and the equilibrium is general.

Notice that in configuration (a, a) of the last lemma, agents can make infinitely many allocations to risky securities resulting in the same risk exposure. These allocations include those which are binding in one of the risky securities.

In the described states of equilibrium, two different types of allocations by investors are possible. With the first type, none of the constraints has to be binding for any investor; and so the instantaneous market price of risk is the same for both risky securities. With the second type of allocations, a pessimist is at the stock short–sale bound, while a realist is at the put short–sale bound. The latter wants to increase his risk exposure by buying more stock, but the former cannot sell it. Consequently, for the stock market to clear, the stock price should rise and its instantaneous market price for risk fall. Similarly, a pessimistic investor wants to take a negative risk exposure by buying more options, but a realist cannot sell them. Thus, to make the options market clear, the put price should rise and its instantaneous market price for risk increase. This mechanism generates significant overpricing of put options in certain states of the economy.

Notice that adding a European call option to the market will not affect our main results. In particular, the put option will still be overpriced. Moreover, the call option will provide the same risk exposure as the stock and will not be mispriced.

More detailed analysis of the constructed partial equilibrium is available in BC for the case of constraints on a short position in the stock and a long position in the derivative.

The characterization of the equilibrium given in Lemma 2 is not complete, since the volatility σ_P is unknown. It follows from Lemma 2 that if condition (22) holds, then the equilibrium admits states with a difference in the instantaneous market prices of risk. However, since this condition explicitly depends on the unknown volatility of the derivative return, the existence of states with a difference in the instantaneous market prices of risk in

a general equilibrium can be verified only after completion of its construction and finding σ_P .

4.1 Pricing Derivative

Let us suppose that the market under consideration is Markovian, so that there exists an m -dimensional state process $Y(t)$ such that

$$dY(t) = \mu(Y, t)dt + \sigma(Y, t)dW(t), \quad Y(0) = Y_0, \quad (31)$$

where μ is an m -dimensional drift and σ is an $m \times 2$ -volatility matrix.

The volatility of the derivative return will be determined after finding the put price as a function of the state variables of the economy. This volatility allows us to complete the characterization of a general equilibrium. We assume from here onwards that the process μ_D^2 and its variance $v(t)$ are given by equations from Appendix A, which are derived in the assumption that the conditional variance $v(0)$ is constant:

$$d\mu_D^2(t) = [a_0 + a_1\mu_D^2(t)]dt + \frac{v(t)}{\sigma_D} dw^2(t), \quad (32)$$

$$v(t) = \frac{\alpha_1 - M\alpha_2 \exp\left[\frac{(\alpha_2 - \alpha_1)t}{\sigma_D^2}\right]}{1 - M \exp\left[\frac{(\alpha_2 - \alpha_1)t}{\sigma_D^2}\right]}, \quad (33)$$

where

$$\alpha_{1,2} = \sigma_D^2 \left(a_1 \mp \sqrt{a_1^2 + \left(\frac{d}{\sigma_D}\right)^2} \right), \quad M = \frac{v(0) - \alpha_1}{v(0) - \alpha_2}. \quad (34)$$

The results from the previous section reveal that the state variable vector of the economy is $Y^T = (Y_1, Y_2, Y_3, Y_4) \equiv (D, \lambda, \mu_D^2, \mu_D)$. Furthermore, the derivative price should be a deterministic function of t and Y : $P = P(Y, t)$, where we set $P \in C^{2,1}$. Hence, the volatility of the derivative return is a continuous function given by

$$\sigma_P = \frac{\sum_{i=1}^4 \sigma_i P_{Y_i}}{P}, \quad (35)$$

where σ_i is the volatility of processes Y_i , $i \in \{1, 2, 3, 4\}$.

The following lemma formulates the problem which has to be solved to find the price of the derivative.

Lemma 3 *If the quasi-linear PDE (36) with the terminal condition (37) is uniformly parabolic, then its solution is equal to the put price $P(Y, t)$:*

$$0 = P_t - rP + \frac{1}{2} \sum_{i,j=1}^3 \sigma_i \sigma_j P_{Y_i Y_j} + \sum_{i=1}^3 P_{Y_i} \left(\mu_i - \sigma_i \frac{\mu_D - r}{\sigma_D} \right) - \Delta \sum_{i,j=1}^3 \sigma_i P_{Y_i} + \frac{1}{2} \sigma_4^2 P_{Y_4 Y_4} + P_{Y_4} \mu_4, \quad (36)$$

$$P(Y, T^*) = [K - D(T - T^*)]^+, \quad (37)$$

where μ_i is the drift of Y_i , $i \in \{1, 2, 3, 4\}$. In addition, $\mu_1 = \mu_D$, $\mu_3 = a_0 + (a_1 - \frac{v}{\sigma_D})\mu_3 + \frac{v}{\sigma_D^2}\mu_4$, $\mu_4 = a_0 + a_1\mu_4$, $\sigma_1 = \sigma_D$, $\sigma_3 = v/\sigma_D$, $\sigma_4 = d$, and coefficients r , μ_2 , σ_2 and Δ take the following values:

If

$$\frac{\lambda + 1}{\lambda} [\sigma_D \lambda (\beta + 1) - \sigma_P \gamma] < \bar{\mu}_D \quad (38)$$

then $r = \mu_D - \sigma_D^2 - \sigma_D [\sigma_D (\beta + 1) \lambda - \gamma \sigma_P]$, $\mu_2 = \Delta \gamma \sigma_P (1 + \lambda)$, $\sigma_2 = \lambda (\Delta - \bar{\mu}_D)$, and $\Delta = \bar{\mu}_D - \frac{1+\lambda}{\lambda} [(\beta + 1) \sigma_D \lambda - \gamma \sigma_P]$;

If inequality (38) does not hold, then $r = \frac{\mu_D^{1+\lambda} \mu_D^2}{1+\lambda} - \sigma_D^2$, $\mu_2 = 0$, $\sigma_2 = -\lambda \bar{\mu}_D$, and $\Delta = 0$.

To evaluate the price of the derivative, we have to solve PDE (36). Unfortunately, finding its analytical solution is an unfeasible task for a European put option. Thus, we resort to the numerical treatment of the problem that will be based on the technique due to Kushner and Depuis (1992).

Finally we notice that instead of a European option we could have considered an American one. As it follows from the above analysis, this would have resulted in quantitative changes

in the equilibrium, but not qualitative ones.

5 Output of the Model

After solving PDE (36), we find the derivative price and the difference in the instantaneous market prices of risk as functions of state variables. For the reason of limited space, we present only the outputs of the model that are directly related to the effect of the put overpricing.

In running our computations, we choose the following values for the parameters: $T = 10$, $T^* = 0.083$, $K = 50$, $\sigma_D = 0.15$, $a_0 = 0.009$, $a_1 = -0.1$, $d = 0.03$, $v(0) = 0.0036$, and $\beta = 0$. One can see that we set σ_D to be close to the volatility of the underlying asset return, rather than to the return of the aggregate consumption (which is empirically only a few percents). This is explained by the relevance for the model in which the stock return volatility plays an important role.¹¹ Moreover, we assume that the drift of the annualized stock return fluctuates around 9% with a mean-reverting rate of 0.1 and a volatility of 3%. Also, agent 2 cannot short the stock and receives a shock at time zero that the drift of the stock return is $\mu_D^2(0)$ with a conditional standard deviation of 0.06 under Q^2 . If the shock is strong and pessimistic, agent 2 does not hold the stock at all. All figures are shown for $t = 0$.

Because $S = D(T - t)$ we show the model outputs versus S rather than D . We compare the equilibrium price of the put option with its benchmark price P_0 found in the assumption that the instantaneous market prices of risk of two risky securities are the same, i.e. $S_P = S_S$,

¹¹According to the recent results by Mankiw, Zeldes (1991), Attanasio, et. al. (2002), and Ait-Sahalia, et. al. (2001), the volatility of consumption growth for stock holders is much higher than that for stock non-holders. Because the volatility of aggregate consumption growth is defined by stock non-holders and we consider only stock holders, our choice of the value of σ_D is not too controversial.

where $S_P = \frac{\mu_P - r}{\sigma_P}$ and $S_S = \frac{\mu_D - r}{\sigma_D}$. We call the states of economy where $P > P_0$ as states with mispricing. Notice that traditionally, the price of an option generated in a model is compared with that of Black and Scholes assuming that the interest rate is constant. This comparison is meaningless in our case, since our model generates a volatile interest rate. Furthermore, we compare our results with those in Table 1. Even though Table 1 describes American options, European options must have very close returns, given their short maturities. In fact, the adjustment for early exercise is negligible for the reported results. Moreover, this table includes the 1987 stock market crash, while our analysis assumes no crashes in the true stock process. Yet, exclusion of this crash from the data does not change the returns and the Sharpe ratio of puts significantly (see Bondarenko (2003)).

Figure 1, Panel A shows the ratio of the instantaneous market prices of risk S_P/S_S when $\mu_D = 0.09$, $\mu_D^2 = -0.20$ (solid line) or $\mu_D^2 = -0.09$ (dashed line) and agents have the same levels of wealth. The ratios are shown for different moneyness K/S . Consistent with empirical results, there is an increasing difference in the instantaneous market prices of risk when the option goes deeper OTM. This difference is due to the type of short-sale constraints on the put option. As seen from equation (23), Δ is higher when $-\gamma\sigma_P$ is small. When the moneyness falls, the price of the put option decreases to zero much faster than its volatility decreases to minus infinity. Thus, since γ is proportional to P , $-\gamma\sigma_P$ decreases with the lower moneyness and the instantaneous market price of risk of the put increases with respect to that of the underlying security. As seen in Panel A, the fall of $-\gamma\sigma_P$ also defines the relative rise of the put instantaneous market price of risk when option is deep ITM. However, this rise is much less significant than that for OTM puts.

Figure 1, Panel B shows the dependence on the moneyness of the conditional return μ_P of the put when $\mu_D = 0.09$, $\mu_D^2 = -0.20$ (solid line) or $\mu_D^2 = -0.09$ (dashed line), and agents

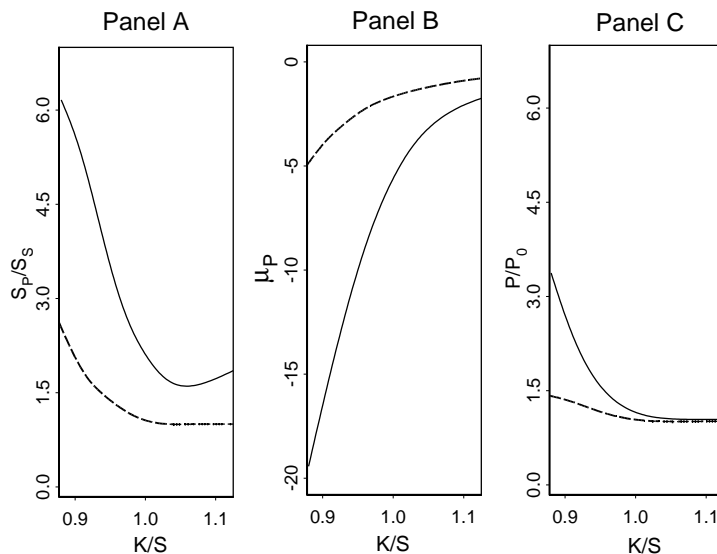


Figure 1: This Figure plots the ratio of the instantaneous market prices of risk S_P/S_S of the put and the stock (Panel A), the conditional return μ_P of the put (Panel B), and the ratio of the prices P/P_0 of the put and that of the put found in the assumption that $S_P = S_S$ (Panel C). The ratios are shown for different moneyness. The time is zero and the state variables are fixed at $\lambda = 1.0$, $\mu_D = 0.09$, $\mu_D^2 = -0.20$ (solid line), and $\mu_D^2 = -0.09$ (dashed line).

have the same levels of wealth. Abnormally high conditional returns of OTM puts follow from their abnormally high instantaneous market prices of risk. We notice that the excessive put returns rising in our model could not be possible if short-sale constraints were absent and $\Delta = 0$. In such a case $\frac{\mu_P - r}{\sigma_P} = S_S$ and $\mu_P = \sigma_P S_S + r$. Given that $S_S \approx 0.6$ for the last fifteen years and σ_P decreases as the put goes OTM, μ_P also decreases for lower moneyness. However its magnitude is far too low: For example, if the put is ATM, then $\sigma_P \approx -4$ and $\mu_P \approx -2.4$, which corresponds to the monthly return of only -0.19 , while the ATM monthly put return on Panel B is -0.40 when $\mu_D^2 = -0.20$. Moreover, the shape of the return would be much less steep.

Figure 1, Panel C plots the ratio of prices of the put options P/P_0 for different moneyness when $\mu_S = 0.09$, $\mu_S^2 = -0.20$ (solid line) or $\mu_S^2 = -0.09$ (dashed line), and agents have the same levels of wealth. Similar to empirical results, the put option becomes more overpriced

when it goes deeper OTM. These results follow from Panel A: When option goes deeper OTM, this means a relatively higher instantaneous market price for risk and thus relatively higher prices. For the same reason, the put option becomes slightly overpriced when it is deep ITM. However, this overpricing is much less pronounced than for that with low moneyness. Panel C also shows that the difference between P and P_0 exists even in states of the economy where $\Delta = 0$. In particular, when the option is ITM and ATM. It follows that if there is a positive probability of Δ becoming positive in finite time, the pessimistic agent increases his demand for the option, hence increasing its price relative to P_0 , even in states where $\Delta = 0$.

The event that $\mu_D^2 = -0.20$ or $\mu_D^2 = -0.09$ is very rare. It is more likely that put prices observed at a given time to the option expiration are taken from the pool of different μ_D^2 . To simplify our interpretation, we assume that $\mu_D^2(0)$ can be equal to -4.0 (the 1987 stock market crash¹²) with the probability p or $\mu_D(0)$ (no shock) with probability $1 - p$.¹³ Figure 2 illustrates mispricing by showing the expected S_P and S_S (Panel A), the expected μ_P (Panel B), and the expected P with expected P_0 (Panel C) versus moneyness at $t = 0$ when $\mu_D(0) = 0.09$ and $\lambda = 1$ (thick lines) or $\lambda = 0.1$ (thin lines). Expectations are taken over the above values of $\mu_D^2(0)$. In the calculation of the expected values, we adjust p to a level at which the expected return of the ATM put at $\lambda = 1$ is equal to that in Table 1. It turns out that matching the expected returns requires that $p = 0.075$. This implies that agent 2 expects the 1987 stock market crash every thirteen months! This probability is at least one order of magnitude greater than one would expect from the historical data. This conclusion favors the irrational behavior of agent 2.¹⁴

¹²We model a stock crash by a continuous process assuming that the stock falls significantly within a month. In the case of the 1987 stock market crash, stock falls by $\approx 30\%$ within a month.

¹³Incorporation of crashes in our model will not change allocations of agent 1 significantly. If agent 2 overestimate probabilities and sizes of crashes then his allocations will change substantially. Thus, we can include crashes in beliefs of agent 2 without assuming them in actual dividend rate.

¹⁴We should caution that $p = 0.075$ is an estimate found from matching the conditional returns averaged

Panel A of Figures 2 shows that the dependence of the instantaneous market price for risk of the put from the moneyness is weak and different from the historical one documented in Table 1 if we assume that $\lambda = 1$ and pessimistic shocks are very strong. Hence, for a better explanation of the put overpricing one has to assume that agent 2 tends to have frequent pessimistic shocks of a relatively small scale if wealths of two agents is approximately the same. If pessimistic shocks are very strong the match between the data in Table 1 and our predictions is considerably better if agent 1 is much wealthier than agent 2 (see lines for $\lambda = 0.1$ in Panel A of Figure 2).¹⁵ Moreover, it seems more likely that investors have strong occasional pessimistic shocks, rather than consistently underestimate stock returns through time. Whether empirical λ is close to 1 or 0.1 remains an open question.

We notice that all results presented in this section for $\lambda = 1$ are very similar to those for $\lambda > 1$ which we do not show.

If our results are averaged over the distribution of $\mu_D^2(0)$, they do not contradict existing market practice when many investors take long positions in both risky securities to obtain portfolio insurance (e.g., see Pan and Poteshman (2003)). Indeed, in states with positive Δ , agent 2 does not hold the stock and has a long position in the put. In states where $\Delta = 0$, the allocations to risky securities cannot be identified uniquely. In these states, agent 2 can find infinitely many long positions in both risky securities such that will provide him with an optimal cumulative risk exposure. We choose one such position for investor 2 and average

over a few states of the economy with historical returns. Accurate comparison requires averaging put prices over all states of the economy and times at a given moneyness, and time to expiration. This comparison is not immediately available since the distribution of state variables is defined by unknown initial values.

¹⁵The intuition behind the difference in the shapes of instantaneous market prices for risk when $\lambda = 1.0$ and $\lambda = 0.1$ is clear: If agent 1 is much wealthier than agent 2, then the former has a strong effect on the behavior of the instantaneous market price for risk of puts. When a pessimistic shock is strong, agent 1 takes the shortest position in the put allowed by the constraint. Because of a strong dependence of this constraint from the put moneyness, the market price for risk of the put has a shape shown in Figure 2. If, however, $\lambda = 1$ and a pessimistic shock is strong, the influence of the shape of the put short-sale constraint on prices is limited.

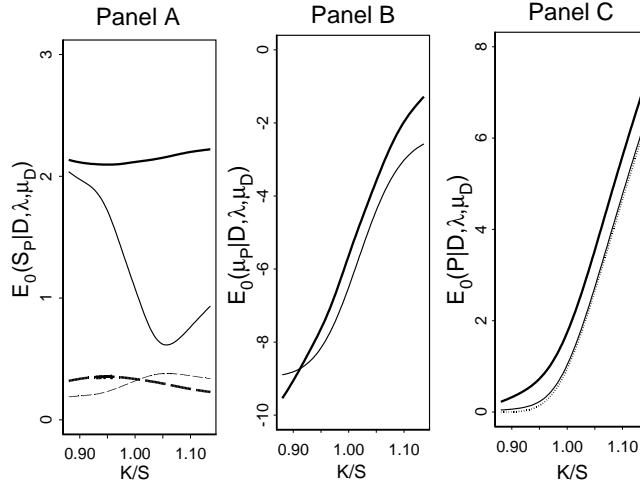


Figure 2: This Figure shows the expected instantaneous market prices of risk (Panel A) of the put (solid line) and the stock (dashed line), the expected conditional returns μ_P of the put (Panel B), and the expected prices (Panel C) of the put (solid line) and that of the put found in the assumption that $S_P = S_S$ (dotted line). The expectation is taken with respect to $\mu_D^2(0)$ that can be equal to -4.00 with a probability of 0.075 and 0.09 with a probability of 0.925. The expectations are shown for different moneyness. The time is zero and the state variables are fixed at $\mu_D = 0.09$ and $\lambda = 1.0$ (thick line), or $\lambda = 0.1$ (thin line).

his allocations over all possible shocks in $\mu_D^2(0)$. The averaging shows that this investor takes long positions in both the put and the stock for all moneyness of the put. This point is illustrated in Figure 3 where we show the cumulative risk exposure of investor 2, as well as his allocations to individual securities in states where $\Delta > 0$ and his possible allocations to individual securities in states where $\Delta = 0$ when $\lambda = 1$ and $K/S = 0.9$. It follows from Figure 3 that agent 2 can optimally choose to take long positions in both risky securities when $\Delta = 0$. Figure 3 is also consistent with the well-known fact that the trading activity in the equity options market is only a tiny fraction of that in the underlying equity market. Indeed, if $X_1 = X_2$, then from the market clearing conditions $\pi_S^1 = 2 - \pi_S^2$ and $\pi_P^1 = -\pi_P^2$. Thus, as it follows from Figure 3, $|\pi_P^2|$ has the order of magnitude of 0.05 while $|\pi_S^2|$ has the order of magnitude of 1 (except for the states with $\Delta > 0$ where $\pi_S^2 = 0$).

Finally, we discuss the implications of the model for the put-call parity violations. We

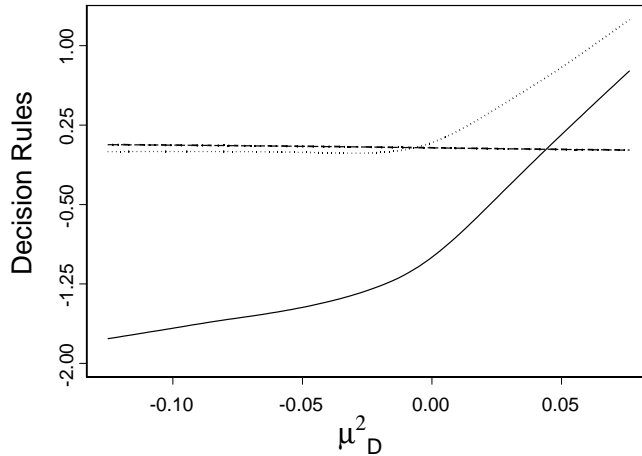


Figure 3: This Figure shows agent 2's decision rules for the cumulative risky asset (solid line), for the stock (dotted line), and for the put (dashed line) at different values of the perceived drift $\mu_D^2(0)$. For states where $\Delta = 0$, the optimal allocations are set to be long in both risky securities. The time is zero and the state variables are fixed at $K/S = 0.9$, $\lambda = 1.0$, and $\mu_D = 0.09$.

notice that our model predicts such violations. Indeed, suppose that B^K is a value of a pure discount bond which pays K dollars at time $T^* \leq T$. Because $B^K(t)\xi_\alpha(t)$ is a Q -martingale for $\alpha = S, P$, B^K is independent from the choice of the asset-specific deflator ξ_α . Then, taking into account that in our economy the European call option C has the same conditional Sharpe ratio as that of the stock, we find the relation

$$S(t) + P(t) = B^K(t) + C(t) + (P(t) - P_0(t)). \quad (39)$$

Therefore, the left side of the last equality is overpriced with respect to $B^K + C$ by $P - P_0$. Even though the ratio P/P_0 increases when the put goes OTM (see Panel C of Figure 1), the difference $P - P_0$ is at its highest when the put is close to ATM. In our model, this difference could be substantial: For example, in Figure 2, the expected difference of put prices is 0.80 when the put is ATM and $\lambda = 1$. If pessimistic shocks decrease in size and increase in frequency so that the expected ATM put return is as in Table 1, then the expected $P - P_0$

decreases significantly. Moreover, if λ is small, then the expected difference $P - P_0$ could be very small even at substantial pessimistic shocks, while the put returns still being very high in magnitude (see Panels B and C of Figure 2 for $\lambda = 0.1$ when the put is ATM). As a result, the put–call parity violations may not be observable because of the bid–ask spreads of securities.¹⁶ This conclusion may indicate that agent 1 is much wealthier than agent 2 if the put–call parity is not violated empirically.

While there are data sets for which violations of the put–call parity have been documented, there are also data sets for which such violations have not been detected within the bid–ask spread. In addition to the argument used above for small λ , the absence of the put–call parity violations and the presence of the put overpricing can be explained as follows. Testing the put–call parity is a common practice in the market aimed at identifying arbitrage opportunities. However this testing cannot determine which particular asset is mispriced: It can say only whether $S + P$ is mispriced with respect to $B^K + C$. If $S + P > B^K + C$, one can try to buy more call options (since long positions are unlimited) to enforce the put–call parity. As a result, overpricing of put options remain, while call options become overpriced, which is observed when calls are ITM.

6 Conclusion

This paper presents a general equilibrium model in which a European put option becomes substantially overpriced when it is deep OTM. The overpricing is due to short-sale constraints on trading the stock and the option as well as the heterogeneity between agents. We

¹⁶When agent 1 is much wealthier than agent 2, his impact on prices is much stronger than that of the other agent. As a result, the put price is very close to P_0 , since this price results when only one agent is present in the economy. We notice that even when $\lambda = 0.1$ agent 2 has a substantial influence on the differential properties of the put, such as its instantaneous market price for risk and returns.

compare the predictions of our model with the empirical findings of Bondarenko (2003) to conclude that their quantitative matching requires a strong assumption about the stock crash expectations by agent 2. This assumption can be relaxed if we combine our approach with that of Bates (2001) or Liu, Pan, and Wang (2003).

Appendix A: Inferencing μ_D^2

This appendix describes the inferencing of μ_D^2 and its variance. The derivation of the drift of the dividend rate D perceived by agent 2 is carried out using filtering theory. We consider the nonlinear filtering of conditionally Gaussian processes, as in Lipster and Shiryaev (1977).

In the filtering problem, agent 2 analyzes two processes:

$$d\mu_D(t) = [a_0 + a_1\mu_D(t)]dt + ddb(t), \quad (40)$$

$$dD(t) = D(t)[\mu_D(t)dt + \sigma_D]dw(t), \quad (41)$$

where D (but not μ_D) is observed.

Since μ_D is not observed, the best that agent 2 can do is to use the posteriori mean $\mu_D^2(t) = E^2[\mu_D(t)|\mathcal{F}_t^D]$. The error of this estimate is characterized by the conditional variance

$$v(t) = E^2[(\mu_D(t) - \mu_D^2(t))^2|\mathcal{F}_t^D].$$

Let us assume that the conditional distribution $Q^2(\mu_D(0) \leq x|D(0))$ is Gaussian, $N(\mu_D^2(0), v(0))$. Then $\mu_D^2(t)$ and $v(t)$ satisfy the equations

$$d\mu_D^2(t) = [a_0 + a_1\mu_D^2(t)]dt + v(t)\frac{1}{\sigma_D}dw^2(t), \quad (42)$$

$$(v(t))' = 2a_1v(t) + d^2 - (v(t))^2\frac{1}{\sigma_D^2}, \quad (43)$$

where $w^2(t)$ is defined by equation (3). The last relations are subject to the initial conditions $\mu_D^2(0) = E^2[\mu_D(0)|D(0)]$, $v(0) = E^2[(\mu_D(0) - \mu_D^2(0))^2|D(0)]$, where, for simplicity of treatment, the random variable $v(0)$ will be set to be equal to a constant.

ODE (43) is the Riccati equation with constant coefficients whose solution is given by

$$v(t) = \frac{\alpha_1 - M\alpha_2 \exp\left[\frac{(\alpha_2 - \alpha_1)t}{\sigma_D^2}\right]}{1 - M \exp\left[\frac{(\alpha_2 - \alpha_1)t}{\sigma_D^2}\right]}, \quad (44)$$

where

$$\alpha_{1,2} = \sigma_D^2 \left(a_1 \mp \sqrt{a_1^2 + \left(\frac{d}{\sigma_D}\right)^2} \right), \quad M = \frac{v(0) - \alpha_1}{v(0) - \alpha_2}. \quad (45)$$

It follows that the process $v(t)$ is deterministic and should be excluded from the list of state variables in the economy.

Appendix B: Proof of Lemma 3

Equation (36) follows from Lemma 2 and the assumption that the economy is markovian. This equation has a solution from Theorem IV.3.1 in Fleming and Soner (1993), since the terminal value $(K - S)^+$ of P is bounded from above and below. Q.E.D.

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